Mimicking the marginal distributions of a semimartingale

Amel Bentata, Rama Cont

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Amel Bentata[1] and Rama Cont[1,2]

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Abstract

We exhibit conditions under which the flow of marginal distributions of a discontinuous semimartingale ξ can be matched by a Markov process, whose infinitesimal generator is expressed in terms of the local characteristics of ξ. Our construction applies to a large class of semimartingales, including smooth functions of a Markov process. We use this result to derive a partial integro-differential equation for the one-dimensional distributions of a semimartingale, extending the Kolmogorov forward equation to a non-Markovian setting.

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Contents

*(1) Laboratoire de Probabilités et Modèles Aléatoires, UMR 7599 CNRS - Université de Paris VI-VII. (2) IEOR Department, Columbia University, New York.

1 Introduction

Stochastic processes with path-dependent / non-Markovian dynamics used in various fields such as physics and mathematical finance present challenges for computation, simulation and estimation. In some applications where one is interested in the marginal distributions of such processes, such as option pricing or Monte Carlo simulation of densities, the complexity of the model can be greatly reduced by considering a low-dimensional Markovian model with the same marginal distributions. Given a process ξ, a Markov process X is said to *mimick* ξ on the time interval $[0,T]$, $T>0$, if ξ and X have the same marginal distributions:

$$\forall t \in [0,T], \qquad \xi_t \stackrel{d}{=} X_t. \tag{1}$$

X is called a *Markovian projection* of ξ. The construction of Markovian projections was first suggested by Brémaud [4] in the context of queues. Construction of mimicking processes of 'Markovian' type has been explored for Ito processes [12] and marked point processes [7]. A notable application is the derivation of forward equations for option pricing [3, 9].

We proposer in this paper a systematic construction of such Markovian projections for (possibly discontinuous) semimartingales. Given a semimartingale ξ, we give conditions under which there exists a Markov process X whose marginal distributions are identical to those of ξ, and give an explicit construction of the Markov process X as the solution of a martingale problem for an integro-differential operator [2, 19, 23, 24].

In the martingale case, the Markovian projection problem is related to the problem of constructing martingales with a given flow of marginals, which dates back to Kellerer [18] and has been recently explored by Yor and coauthors [1, 14, 20] using a variety of techniques. The construction proposed in this paper is different from the does not rely on the martingale property of ξ. We shall see nevertheless that our construction preserves the (local) martingale property. Also, whereas the approaches described in [1, 14, 20] use as a starting point the marginal distributions of ξ, our construction describes the mimicking Markov process X in terms of the local characteristics [16] of the semimartingale ξ. Our construction thus applies more readily to solutions of stochastic differential equations where the local characteristics are known but not the marginal distributions.

Section 2 presents a Markovian projection result for a $\mathbb{R}^d$-valued semimartingale given by its local characteristics. We use these results in section 2.4 to derive a partial integro-differential equation for the one-dimensional distributions of a discontinuous semimartingale, thus extending the Kolmogorov forward equation to a non-Markovian setting. Section 3 shows how this result may be applied to processes whose jumps are represented as the integral of a predictable jump amplitude with respect to a Poisson random measure, a representation often used in stochastic differential equations with jumps. In Section 4 we show that our construction applied to a large class of semimartingales, including smooth functions of a Markov process (Section 4.1), and time-changed Lévy processes

(Section 4.2).

2 A mimicking theorem for discontinuous semimartingales

Consider, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, an Ito semimartingale, on the time interval $[0,T]$, $T>0$, given by the decomposition

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int_{\|y\|\leq 1} y\, \tilde{M}(ds\, dy) + \int_0^t \int_{\|y\|>1} y\, M(ds\, dy), \tag{2}$$

where ξ_0 is in $\mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, M is an integer-valued random measure on $[0,T]\times\mathbb{R}^d$ with compensator measure μ and $\tilde{M} = M - \mu$ is the compensated measure [16, Ch.II,Sec.1], β (resp. δ) is an adapted process with values in $\mathbb{R}^d$ (resp. $M_{d\times n}(\mathbb{R})$).

Our goal is to construct a Markov process, on some filtered probability space $(\Omega_0, \mathcal{B}, (\mathcal{B}_t)_{t\geq 0}, \mathbb{Q})$ such that X and ξ have the same marginal distributions on $[0,T]$, i.e. the law of X_t under $\mathbb{Q}$ coincides with the law of ξ_t under $\mathbb{P}$. We will construct X as the solution to a *martingale problem* [11, 23, 25, 21] on the canonical space $\Omega_0 = D([0,T],\mathbb{R}^d)$.

2.1 Martingale problems for integro-differential operators

Let $\Omega_0 = D([0,T],\mathbb{R}^d)$ be the Skorokhod space of right-continuous functions with left limits. Denote by $X_t(\omega) = \omega(t)$ the canonical process on Ω_0, $\mathcal{B}^0_t$ its filtration and $\mathcal{B}_t \equiv \mathcal{B}^0_{t+}$. Our goal is to construct a probability measure $\mathbb{Q}$ on Ω_0 such that X is a Markov process under $\mathbb{Q}$ and ξ and X have the same one-dimensional distributions:

$$\forall t \in [0,T], \quad \xi_t \stackrel{\text{d}}{=} X_t.$$

In order to do this, we shall characterize $\mathbb{Q}$ as the solution of a *martingale problem* for an appropriately chosen integro-differential operator $\mathcal{L}$.

Let $\mathcal{C}^0_b(\mathbb{R}^d)$ denote the set of bounded and continuous functions on $\mathbb{R}^d$ and $\mathcal{C}^\infty_0(\mathbb{R}^d)$ the set of infinitely differentiable functions with compact support on $\mathbb{R}^d$. Consider a time-dependent integro-differential operator $\mathcal{L} = (\mathcal{L}_t)_{t\in[0,T]}$ defined, for $f \in \mathcal{C}^\infty_0(\mathbb{R}^d)$, by

$$\begin{aligned}\mathcal{L}_t f(x) &= b(t,x).\nabla f(x) + \sum_{i,j=1}^d \frac{a_{ij}(t,x)}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ &+ \int_{\mathbb{R}^d} [f(x+y) - f(x) - 1_{\{\|y\|\leq 1\}}\, y.\nabla f(x)] n(t,dy,x),\end{aligned} \tag{3}$$

where $a : [0,T]\times\mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$, $b : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ are measurable functions and, for each $(n(t,\,.\,,x), (t,x) \in [0,T]\times\mathbb{R}^d)$ is a measurable family of positive measures on $\mathbb{R}^d - \{0\}$.

For x_0 in $\mathbb{R}^d$, we recall that a probability measure $\mathbb{Q}_{x_0}$ on $(\Omega_0, \mathcal{B}_T)$ is a solution to the *martingale problem* for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$ if $\mathbb{Q}\left(X_0 = x_0\right) = 1$ and for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, the process

$$f(X_t) - f(x_0) - \int_0^t \mathcal{L}_s f(X_s)\, ds$$

is a $(\mathbb{Q}_{x_0}, (\mathcal{B}_t)_{t\geq 0})$-martingale on $[0,T]$. Existence, uniqueness and regularity of solutions to martingale problems for integro-differential operators have been studied under various conditions on the coefficients [25, 15, 11, 19, 21].

We make the following assumptions on the coefficients:

Assumption 1 (Boundedness of coefficients)**.** *There exists $K_1 > 0$*

$$\forall (t,z) \in [0,T] \times \mathbb{R}^d \quad \|b(t,z)\| + \|a(t,z)\| + \int \left(1 \wedge \|y\|^2\right)\, n(t,dy,z) \leq K_1$$

and

$$\lim_{R\to\infty} \int_0^T \sup_{z\in\mathbb{R}^d} n\left(t, \{\|y\| \geq R\}, z\right)\, dt = 0.$$

where $\|.\|$ denotes the Euclidean norm.

Assumption 2 (Continuity)**.** *For all $t \in [0,T]$ and $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$, $b(t,.)$, $a(t,.)$ and , $n(t,B,.)$ are continuous on $\mathbb{R}^d$, uniformly in $t \in [0,T]$.*

Assumption 3 (Non-degeneracy)**.**

Either $\quad \forall R > 0\, \forall t \in [0,T] \quad \inf_{\|z\|\leq R} \inf_{x\in\mathbb{R}^d} {}^t x.a(t,z).x > 0$

or $\quad a \equiv 0$ and there exists$\beta \in]0,2[, C > 0$, and a family $n^\beta(t,dy,z)$ of positive measures such that

$$\forall (t,z) \in [0,T] \times \mathbb{R}^d \qquad n(t,dy,z) = n^\beta(t,dy,z) + \frac{C}{\|y\|^{d+\beta}}\, dy,$$

$$\int \left(1 \wedge \|y\|^\beta\right)\, n^\beta(t,dy,z) \leq K_2, \quad \lim_{\epsilon\to 0} \sup_{z\in\mathbb{R}^d} \int_{\|y\|\leq\epsilon} \|y\|^\beta\, n^\beta(t,dy,z) = 0.$$

Mikulevicius and Pragarauskas [21] show that if $\mathcal{L}$ satisfies Assumptions 1–3 (in which corresponds to a "non-degenerate Lévy operator" in the terminology of [21]) the martingale problem for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d)$) has a unique solution for every initial condition $x_0 \in \mathbb{R}^d$:

Proposition 1. *[21, Theorem 5]*
Under Assumptions 1, 2 and 3 the martingale problem for $((\mathcal{L}_t)_{t\in[0,T]}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$ is well-posed : for any $x_0 \in \mathbb{R}^d$,there exists a unique probability measure $\mathbb{Q}_{x_0}$ on $(D([0,T],\mathbb{R}^d), \mathcal{F}_T)$ such that $\mathbb{Q}_{x_0}(X_0 = x_0) = 1$ and for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$$f(X_t) - f(x_0) - \int_0^t \mathcal{L}_s f(X_s)\, ds$$

is a $\mathbb{Q}_{x_0}$-martingale. Under $\mathbb{Q}_{x_0}$, (X_t) is a Markov process and the evolution operator $(Q_t)_{t\in[0,T]}$ defined by

$$\forall f \in \mathcal{C}_b^0(\mathbb{R}^d) \quad Q_t f(x_0) = \mathbb{E}^{\mathbb{Q}_{x_0}}\left[f(X_t)\right] \tag{4}$$

is strongly continuous on $[0,T[$.

2.2 A uniqueness result for the Kolmogorov forward equation

An important property of continuous-time Markov processes is their link with partial (integro-)differential equation (PIDE) which allows to use analytical tools for studying their probabilistic properties. In particular the transition density of a Markov process solves the forward Kolmogorov equation (or Fokker-Planck equation) [24]. The following result shows that under Assumptions 1, 2 and 3 the forward equation corresponding to $\mathcal{L}$ has a unique solution:

Theorem 1 (Kolmogorov Forward equation)**.** *Under Assumptions 1, 2 and 3, each x_0 in $\mathbb{R}^d$, there exists a unique family $(p_t(x_0,dy), t\in[0,T])$ of bounded measures on $\mathbb{R}^d$ such that $p_0(x_0,.) = \epsilon_{x_0}$ is the point mass at x_0 and*

$$\forall g \in \mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R}), \qquad \int g(y)\frac{dp}{dt}(x_0,dy) = \int p_t(x_0,dy)L_t g(y). \tag{5}$$

$p_t(x_0,.)$ is the marginal distribution at time t of the unique solution associated to the martingale problem for $(\mathcal{L},\mathcal{C}_0^\infty(\mathbb{R}^d)\ )$ starting from x_0 on $[0,T]$.

Proof. Under Assumptions 1, 2 and 3 Proposition 1 implies that the martingale problem for $\mathcal{L}$ on the domain $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is well-posed. For any x_0 in $\mathbb{R}^d$, denote $(X,\mathbb{Q}_{x_0})$ the unique solution of the martingale problem for $\mathcal{L}$. Define

$$\forall t\in[0,T] \quad \forall f \in \mathcal{C}_b^0(\mathbb{R}^d) \quad Q_t f(x_0) = \mathbb{E}^{\mathbb{Q}_{x_0}}\left[f(X_t)\right]. \tag{6}$$

Q_t is the evolution operator on $[0,T]$ of X on $\mathcal{C}_b^0(\mathbb{R}^d)$. $(Q_t)_{t\in[0,T]}$ is then strongly continuous on $[0,T[$. If $q_t(x_0,dy)$ denotes the law of (X_t) under $\mathbb{Q}_{x_0}$, the martingale property shows that $q_t(x_0,dy)$ satisfies the equation (5). Integration of (5) with respect to time yields

$$\int q_t(x_0,dy)g(y) = g(x_0) + \int_0^t \int q_s(x_0,dy)\mathcal{L}_s g(y)\,ds. \tag{7}$$

We have thus constructed one solution q_t of (5) with initial condition $q_0(dy) = \epsilon_{x_0}$. This solution of (5) is in particular positive with mass 1.

To show uniqueness of solutions of (5), we will rewrite (5) as the forward Kolmogorov equation associated with a *homogeneous* operator on space-time domain and use uniqueness results for the corresponding homogeneous equation. Let $f\in C_0^\infty(\mathbb{R}^d)$ and $\gamma\in\mathcal{C}^1([0,T])$ and consider the (homogeneous) dependent operator A mapping functions of the form $(t,x) \in [0,T]\times\mathbb{R}^d \to f(x)\gamma(t)$, which will be denoted $\mathcal{C}^1([0,T])\otimes\mathcal{C}_0^\infty(\mathbb{R}^d)$, into :

$$A(f\gamma)(t,x) = \gamma(t)\mathcal{L}_t f(x) + f(x)\gamma'(t). \tag{8}$$

For any x_0 in $\mathbb{R}^d$, if $(X,\mathbb{Q}_{x_0})$ is a solution of the martingale problem $\mathcal{L}$, then the law of $\eta_t = (t, X_t)$ is a solution of the martingale problem for A: for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $\gamma \in \mathcal{C}([0,T])$,

$$\int q_t(x_0,dy)f(y)\gamma(t) = f(x_0)\gamma(0) + \int_0^t \int q_s(x_0,dy)A(f\gamma)(s,y)\,ds. \tag{9}$$

Using [11, Theorem 7.1 and Theorem 10.1,Chapter 4]), uniqueness holds for the martingale problem associated to the operator $\mathcal{L}$ on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ if and only if uniqueness holds for the martingale problem associated to the martingale problem for A on $\mathcal{C}^1([0,T])\otimes C_0^\infty(\mathbb{R}^d)$.
Define for all t in $[0,T]$, for all g in $\mathcal{C}^1([0,T])\otimes\mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$\mathcal{Q}_t g(x_0) = \int_{\mathbb{R}^d} q_t(x_0,dy)g(t,y) \tag{10}$$

One observes that $\mathcal{Q}_t.(x_0)$ corresponds to the extension of Q_t defined on the domain $\mathcal{C}_0^\infty(\mathbb{R}^d)$ to the domain $\mathcal{C}^1([0,T])\otimes\mathcal{C}_0^\infty(\mathbb{R}^d)$.
Consider now a family $p_t(dy)$ of positive measures solution of (7) with $p_0(dy) = \epsilon_{x_0}(dy)$. After integration by parts

$$\int_{\mathbb{R}^d} p_t(dy)f(y)\gamma(t) = f(x_0)\gamma(0) + \int_0^t \int_{\mathbb{R}^d} p_s(dy)A(f\gamma)(s,y)\,ds \tag{11}$$

holds true. Define for all t in $[0,T]$, for all g in $\mathcal{C}^1([0,T])\otimes\mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$P_t g \quad = \quad \int_{\mathbb{R}^d} p_t(dy)g(t,y).$$

Given (7) and (11), for all $\epsilon > 0$:

$$\begin{aligned}\mathcal{Q}_t(f\gamma)(x_0) - \mathcal{Q}_\epsilon(f\gamma)(x_0) &= \int_\epsilon^t \int_{\mathbb{R}^d} q_u(x_0,dy)A(f\gamma)(u,y)\,du = \int_\epsilon^t Q_u(A(f\gamma))(x_0)\,du,\\ P_t(f\gamma) - P_\epsilon(f\gamma) &= \int_\epsilon^t \int_{\mathbb{R}^d} p_u(dy)A(f\gamma)(u,y)\,du = \int_\epsilon^t P_u(A(f\gamma))\,du.\end{aligned} \tag{12}$$

For any $\lambda > 0$, we have

$$\begin{aligned}\lambda\int_0^\infty e^{-\lambda t}\,\mathcal{Q}_t(f\gamma)(x_0)\,dt &= f(x_0)\gamma(0) + \lambda\int_0^\infty e^{-\lambda t}\int_0^t \mathcal{Q}_s(A(f\gamma))(x_0)\,ds\,dt\\ &= f(x_0)\gamma(0) + \lambda\int_0^\infty e^{-\lambda t}\left(\int_s^\infty e^{-\lambda t}dt\right)\mathcal{Q}_s(A(f\gamma))(x_0)\,ds\\ &= f(x_0)\gamma(0) + \int_0^\infty e^{-\lambda s}\,\mathcal{Q}_s(A(f\gamma))(x_0)\,ds\end{aligned}$$

Consequently,

$$\int_0^\infty e^{-\lambda t}\, \mathcal{Q}_t(\lambda - A)(f\gamma)(0,x_0)\, dt = f(x_0)\gamma(0) = \int_0^\infty e^{-\lambda t}\, P_t(\lambda - A)(f\gamma)\, dt. \tag{13}$$

$\mathcal{Q}_t$ defines a strongly continuous semigroup on $\mathcal{C}_b^0([0,T]\times\mathbb{R}^d)$. The Hille-Yosida theorem [11, Proposition 2.1 and Theorem 2.6] then implies that for all $\lambda > 0$, $\mathcal{R}(\lambda - A) = \mathcal{C}_b^0([0,T]\times\mathbb{R}^d)$, where $\mathcal{R}(\lambda - A)$ denotes the image of $\mathcal{C}^1([0,T])\otimes \mathcal{C}_0^\infty(\mathbb{R}^d)$ by the mapping $g \to (\lambda - A)g$. Hence, since (13) holds then for all h in $\mathcal{C}_b^0([0,T]\times\mathbb{R}^d)$

$$\int_0^\infty e^{-\lambda t}\, \mathcal{Q}_t h\,(0,x_0)\, dt = \int_0^\infty e^{-\lambda t}\, P_t h\, dt \tag{14}$$

so the Laplace transform of $t \mapsto \mathcal{Q}_t h\,(0,x_0)$ is uniquely determined. We will now show that $t \mapsto \mathcal{Q}_t h\,(0,x_0)$ is right-continuous. Furthermore $t \to b(t,\,.)$, $t \to a(t,\,.)$, and $t \to n(t,\,.\,,\,.)$ are bounded in t on $[0,T]$ (Assumption 1). It implies that for any fixed $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and any fixed $\gamma \in \mathcal{C}^1([0,T])$, $t \to \mathcal{Q}_t A(f\gamma)(x_0)$ and $t \to P_t A(f\gamma)$ are bounded on $[0,T]$. [11, Theorem 2.6] implies also that the set $\{A(f\gamma), f \in \mathcal{C}_0^\infty(\mathbb{R}^d), \gamma \in \mathcal{C}^1([0,T])\}$ is dense in $\mathcal{C}_b^0([0,T]\times\mathbb{R}^d)$ and (12) shows that $\mathcal{Q}_t.(x_0)$ and $P_t.$ are weakly right-continuous in t on $\mathcal{C}^1([0,T])\otimes\mathcal{C}_0^\infty(\mathbb{R}^d)$, i.e, for $T \geq t' \geq t$:

$$\lim_{t'\to t} P_{t'}(f\gamma) = P_t(f\gamma) \qquad \lim_{t'\to t} \mathcal{Q}_{t'}(f\gamma)(x_0) = \mathcal{Q}_t(f\gamma)(x_0).$$

Since $\mathcal{C}_b^0([0,T]\times\mathbb{R}^d)$ is separating [11, Proposition 4.4, Chapter 3], $\mathcal{Q}_t.(x_0)$ and $P_t.$ are weakly right-continuous and (14) holds for any $\lambda > 0$, we have $\int h(y)q_t(x_0,dy) = \int h(y)p_t(dy)$ for all $h \in \mathcal{C}_b^0(\mathbb{R}^d)$. This ends the proof. □

Remark 2.1. *Assumptions 1, 2 and 3 are sufficient but not necessary for the well-posedness of the martingale problem. For example, the boundedness Assumption 1 may be relaxed to local boundedness, using localization techniques developed in [23, 25]. Such extensions are not trivial and, in the unbounded case, additional conditions are needed to ensure that X does not explode (see [25, Chapter 10]).*

2.3 Markovian projection of a semimartingale

We will make the following assumptions, which are almost-sure analogs of Assumptions 1, 2 and 3, on the local characteristics of the semimartingale ξ:

Assumption 4. *β, δ are bounded on $[0,T]$:*

$$\exists K_1 > 0, \forall t \in [0,T] \;\; \|\beta_t\| \leq K_1, \;\; \|\delta_t\| \leq K_1 \qquad \text{a.s.}$$

Assumption 5. *μ has a density $m(\omega, t, dy)$ with respect to the Lebesgue measure on $[0, T]$ which satisfies*

$$\exists K_2 > 0, \forall t \in [0,T] \quad \int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \, m(., t, dy) \leq K_2 < \infty \qquad \text{a.s.}$$

$$\text{and} \quad \lim_{R\to\infty} \int_0^T m\left(., t, \{\|y\| \geq R\}\right) \, dt = 0 \quad \text{a.s.}$$

Assumption 6.

Either (i) $\exists \epsilon > 0, \forall t \in [0, T[\ {}^t\delta_t\delta_t \geq \epsilon \ I_d$ a.s.

or (ii) $\delta \equiv 0$ and there exists $\beta \in]0, 2[, c, K_3 > 0$, and a family $m^\beta(t, dy)$ of positive measures such that

$$\forall t \in [0, T[\qquad m(t, dy) = m^\beta(t, dy) + \frac{c}{\|y\|^{d+\beta}} \, dy \text{ a.s.},$$

$$\int \left(1 \wedge \|y\|^\beta\right) \, m^\beta(t, dy) \leq K_3, \quad \lim_{\epsilon\to 0} \int_{\|y\|\leq\epsilon} \|y\|^\beta \, m^\beta(t, dy) = 0 \text{ a.s.}$$

Note that Assumption 5 is only slightly stronger than stating that m is a Lévy kernel since in that case we already have $\int \left(1 \wedge \|y\|^2\right) \, m(., t, dy) < \infty$. Assumption 6 extends the "ellipticity" assumption to the case of pure-jump semimartingales and holds for a large class of semimartingales driven by stable or tempered stable processes.

Theorem 2 (Markovian projection)**.** *Define, for $(t, z) \in [0, T] \times \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d - \{0\})$,*

$$\begin{aligned} b(t, z) &= \mathbb{E}\left[\beta_t | \xi_{t^-} = z\right], \\ a(t, z) &= \mathbb{E}\left[{}^t\delta_t\delta_t | \xi_{t^-} = z\right], \\ n(t, B, z) &= \mathbb{E}\left[m(., t, B) | \xi_{t^-} = z\right]. \end{aligned} \tag{15}$$

If (β, δ, n) satisfies Assumptions 4, 5, 6 and (b, a, n) satisfies Assumption 2 then there exists a Markov process $((X_t)_{t\in[0,T]}, \mathbb{Q}_{\xi_0})$, with infinitesimal generator $\mathcal{L}$ defined by (3)*, whose marginal distributions mimick those of ξ:*

$$\forall t \in [0, T] \quad X_t \overset{\mathrm{d}}{=} \xi_t.$$

X is the weak solution of the stochastic differential equation

$$\begin{aligned} X_t = \xi_0 &+ \int_0^t b(u, X_u) \, du + \int_0^t \Sigma(u, X_u) \, dB_u \\ &+ \int_0^t \int_{\|y\|\leq 1} y \, \tilde{N}(du \ dy) + \int_0^t \int_{\|y\|>1} y \, N(du \ dy), \end{aligned} \tag{16}$$

where (B_t) is an n-dimensional Brownian motion, N is an integer-valued random measure on $[0, T] \times \mathbb{R}^d$ with compensator $n(t, dy, X_{t-}) \, dt$, $\tilde{N} = N - n$ the associated compensated random measure and $\Sigma \in C^0([0, T] \times \mathbb{R}^d, M_{d\times n}(\mathbb{R}))$ such that ${}^t\Sigma(t, z)\Sigma(t, z) = a(t, z)$.

We will call $(X, \mathbb{Q}_{\xi_0})$ the *Markovian projection* of ξ.

Proof. First, we observe that n is a Lévy kernel : for any $(t,z) \in [0,T] \times \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) n(t,dy,z) = \mathbb{E}\left[\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) m(t,dy) | \xi_{t^-} = z\right] < \infty \text{ a.s.},$$

using Fubini's theorem and Assumption 5. Consider now the case of a pure jump semimartingale verifying (ii) and define, for $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$,

$$\forall z \in \mathbb{R}^d \quad n^\beta(t,B,z) = \mathbb{E}\left[\int_B m(t,dy,\omega) - \frac{c\,dy}{\|y\|^{d+\beta}} | \xi_{t^-} = z\right].$$

As argued above, n^β is a Lévy kernel on $\mathbb{R}^d$. Assumptions 4 and 5 imply that (b,a,n) satisfies Assumption 1. Furthermore, under assumptions either (i) or (ii) for (δ, m), Assumption 3 holds for (b,a,n). Together with Assumption 2 yields that $\mathcal{L}$ is a non-degenerate operator and Proposition 1 implies that the martingale problem for $(\mathcal{L}_t)_{t\in[0,T]}$ on the domain $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is well-posed. Denote $((X_t)_{t\in[0,T]}, \mathbb{Q}_{\xi_0})$ its unique solution starting from ξ_0 and $q_t(\xi_0, dy)$ the marginal distribution of X_t.

Let f in $\mathcal{C}_0^\infty(\mathbb{R}^d)$. Itô's formula yields

$$\begin{aligned}
f(\xi_t) &= f(\xi_0) + \sum_{i=1}^d \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\xi_{s^-})\, d\xi_s^i + \frac{1}{2}\int_0^t \operatorname{tr}\left[\nabla^2 f(\xi_{s^-})\, {}^t\delta_s \delta_s\right] ds \\
&+ \sum_{s\le t}\left[f(\xi_{s^-} + \Delta\xi_s) - f(\xi_{s^-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\xi_{s^-})\Delta\xi_s^i\right] \\
&= f(\xi_0) + \int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds + \int_0^t \nabla f(\xi_{s^-}).\delta_s dW_s \\
&+ \frac{1}{2}\int_0^t \operatorname{tr}\left[\nabla^2 f(\xi_{s^-})\, {}^t\delta_s \delta_s\right] \; ds \\
&+ \int_0^t \int_{\|y\|\le 1} \nabla f(\xi_{s^-}).y\, \tilde{M}(ds\, dy) \\
&+ \int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s^-} + y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right) M(ds\, dy).
\end{aligned}$$

We note that

- since $\|\nabla f\|$ is bounded $\int_0^t \int_{\|y\|\le 1} \nabla f(\xi_{s^-}).y\, \tilde{M}(ds\, dy)$ is a square-integrable martingale.
- $\int_0^t \int_{\|y\|>1} \nabla f(\xi_{s^-}).y\, M(ds\, dy) < \infty$ a.s. since $\|\nabla f\|$ is bounded.
- since $\nabla f(\xi_{s^-})$ and δ_s are uniformly bounded on $[0,T]$, $\int_0^t \nabla f(\xi_{s^-}).\delta_s dW_s$ is a martingale.

Hence, taking expectations, we obtain:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[f(\xi_t)\right] &= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds\right] + \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})^t \delta_s \delta_s\right]\, ds\right] \\
&+ \mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right)\, M(ds\, dy)\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds\right] + \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\, {}^t\delta_s \delta_s\right]\, ds\right] \\
&+ \mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right)\, m(s,dy)\, ds\right].
\end{aligned}
$$

Observing that:

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds\right] \le \|\nabla f\|\, \mathbb{E}^{\mathbb{P}}\left[\int_0^t \|\beta_s\|\, ds\right] < \infty, \\
&\mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\, {}^t\delta_s \delta_s\right]\right] \le \|\nabla^2 f\|\, \mathbb{E}^{\mathbb{P}}\left[\int_0^t \|\delta_s\|^2\, ds\right] < \infty, \\
&\mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\mathbb{R}^d} \left\|f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right\|\, m(s,dy)\, ds\right] \\
&\quad \le \frac{\|\nabla^2 f\|}{2}\mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\|y\|\le 1} \|y\|^2\, m(s,dy)\, ds\right] + 2\|f\|\mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\|y\|>1} m(s,dy)\, ds\right] < +\infty,
\end{aligned}
$$

we may apply Fubini's theorem to obtain

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[f(\xi_t)\right] &= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \int_0^t \mathbb{E}^{\mathbb{P}}\left[\nabla f(\xi_{s^-}).\beta_s\right]\, ds + \frac{1}{2}\int_0^t \mathbb{E}^{\mathbb{P}}\left[\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\, {}^t\delta_s \delta_s\right]\right]\, ds \\
&+ \int_0^t \mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right)\, m(s,dy)\right]\, ds.
\end{aligned}
$$

Conditioning on ξ_{t-} and using the iterated expectation property,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[f(\xi_t)\right] &= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \int_0^t \mathbb{E}^{\mathbb{P}}\left[\nabla f(\xi_{s^-}).\mathbb{E}^{\mathbb{P}}\left[\beta_s|\xi_{s-}\right]\right]\, ds \\
&+ \frac{1}{2}\int_0^t \mathbb{E}^{\mathbb{P}}\left[\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\, \mathbb{E}^{\mathbb{P}}\left[{}^t\delta_s \delta_s|\xi_{s-}\right]\right]\right]\, ds \\
&+ \int_0^t \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right)\, m(s,dy)|\xi_{s-}\right]\right]\, ds \\
&= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \int_0^t \mathbb{E}^{\mathbb{P}}\left[\nabla f(\xi_{s^-}).b(s,\xi_{s-})\right]\, ds + \frac{1}{2}\int_0^t \mathbb{E}^{\mathbb{P}}\left[\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\, a(s,\xi_{s-})\right]\right]\, ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{P}}\left[\left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right)\, n(s,dy,\xi_{s-})\right]\, ds.
\end{aligned}
$$

Hence

$$
\mathbb{E}^{\mathbb{P}}\left[f(\xi_t)\right] = \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^t \mathcal{L}_s f(\xi_{s-})\, ds\right]. \tag{17}
$$

Let $p_t(dy)$ denote the law of (ξ_t) under $\mathbb{P}$, (17) writes:

$$\int_{\mathbb{R}^d} p_t(dy) f(y) = \int_{\mathbb{R}^d} p_0(dy) f(y) + \int_0^t \int_{\mathbb{R}^d} p_s(dy) \mathcal{L}_s f(y)\, ds. \tag{18}$$

Hence $p_t(dy)$ satisfies the Kolmogorov forward equation (5) for the operator $\mathcal{L}$ with the initial condition $p_0(dy) = \mu_0(dy)$ where μ_0 denotes the law of ξ_0. Applying Theorem 1, the flows $q_t(\xi_0, dy)$ of X_t and $p_t(dy)$ of ξ_t are the same on $[0,T]$. This ends the proof.

□

Remark 2.2 (Mimicking conditional distributions)**.** *The construction in Theorem 2 may also be carried out using*

$$\begin{aligned} b_0(t,z) &= \mathbb{E}\left[\beta_t | \xi_{t^-} = z,\, \mathcal{F}_0\right], \\ a_0(t,z) &= \mathbb{E}\left[{}^t\delta_t \delta_t | \xi_{t^-} = z,\, \mathcal{F}_0\right], \\ n_0(t,B,z) &= \mathbb{E}\left[m(.,t,B) | \xi_{t^-} = z,\, \mathcal{F}_0\right]. \end{aligned}$$

instead of (b,a,n) *in* (15)*. If* (b_0, a_0, n_0) *satisfies Assumption* (3)*, then following the same procedure we can construct a Markov process* $(X, \mathbb{Q}^0_{\xi_0})$ *whose infinitesimal generator has coefficients* (b_0, a_0, n_0) *such that*

$$\forall f \in \mathcal{C}^0_b(\mathbb{R}^d), \forall t \in [0,T] \quad \mathbb{E}^{\mathbb{P}}\left[f(\xi_t)|\mathcal{F}_0\right] = \mathbb{E}^{\mathbb{Q}^0_{\xi_0}}\left[f(X_t)\right],$$

i.e. the marginal distribution of X_t *matches the conditional distribution of* ξ_t *given* $\mathcal{F}_0$*.*

Remark 2.3. *For Ito processes (i.e. continuous semimartingales of the form* (2) *with* $\mu = 0$*), Gyöngy [12, Theorem 4.6] gives a "mimicking theorem" under the non-degeneracy condition* ${}^t\delta_t.\delta_t \geq \epsilon I_d$ *which corresponds to our Assumption 6, but without requiring the continuity condition (Assumption 2) on* (b,a,n)*. Brunick & Shreve [5] extend this result by relaxing the ellipticity condition of [12]. In both cases, the mimicking process* X *is constructed as a weak solution to the SDE* (16) *(without the jump term), but this weak solution does not in general have the Markov property: indeed, it need not even be unique under the assumptions used in [12, 5]. In particular, in the setting used in [12, 5], the law of* X *is not uniquely determined by its 'infinitesimal generator'* $\mathcal{L}$*. This makes it difficult to 'compute' quantities involving* X*, either through simulation or by solving a partial differential equation.*

By contrast, under the additional continuity condition 2 on the projected coefficients, X *is a Markov process whose law is uniquely determined by its infinitesimal generator* $\mathcal{L}$ *and whose marginals are the unique solution of the Kolmogorov forward equation* (5)*. This makes it possible to compute the marginals of* X *by simulating the SDE* (16) *or by solving a forward PIDE.*

It remains to be seen whether the additional Assumption 2 is verified in most examples of interest. We will show in Section 4 that this is indeed the case.

Remark 2.4 (Markovian projection of a Markov process). *The term* Markovian projection *is justified by the following remark: if the semimartingale ξ is already a Markov process and satisfies the assumption of Theorem 2, then the uniqueness in law of the solution to the martingale problem for $\mathcal{L}$ implies that the Markovian projection $(X, \mathbb{Q}_{\xi_0})$ of ξ has the same law as $(\xi, \mathbb{P}_{\xi_0})$. So the map which associates (the law $\mathbb{Q}_{\xi_0}$ of) X to ξ may indeed be viewed as a projection; in particular it is involutive.*

This property contrasts with other constructions of mimicking processes [1, 7, 12, 13, 20] which fail to be involutive. A striking example is the construction, by Hamza & Klebaner [13], of discontinuous martingales whose marginals match those of a Gaussian Markov process.

2.4 Forward equations for semimartingales

Theorem 1 and Theorem 2 allow us to obtain a forward PIDE which extends the Kolmogorov forward equation to semimartingales which verify the Assumptions of Theorem 2:

Theorem 3. *Let ξ be a semimartingale given by* (2) *satisfying the assumptions of Theorem 2. Denote $p_t(dx)$ the law of ξ_t on $\mathbb{R}^d$. $t \mapsto p_t$ is the unique solution, in the sense of distributions, of the forward equation*

$$\forall t \in [0,T] \quad \frac{\partial p_t}{\partial t} = \mathcal{L}_t^\star . \, p_t, \tag{19}$$

with initial condition $p_0 = \mu_0$, where μ_0 denotes the law of ξ_0,
where $\mathcal{L}^\star$ is the adjoint of $\mathcal{L}$, defined by

$$\begin{aligned}
\forall g &\in C_0^\infty(\mathbb{R}^d, \mathbb{R}), \\
\mathcal{L}_t^\star g(x) &= -\nabla \left[b(t,x) g(x) \right] + \nabla^2 \left[\frac{a(t,x)}{2} g(x) \right] \\
&+ \int_{\mathbb{R}^d} \left[g(x-z) n(t,z,x-z) - g(x) n(t,z,x) - 1_{\|z\| \le 1} z . \nabla \left[g(x) n(t,dz,x) \right] \right],
\end{aligned} \tag{20}$$

where the coefficients b, a, n are defined as in (15).

Proof. The existence and uniqueness is a direct consequence of Theorem 1 and Theorem 2. To finish the proof, let compute $\mathcal{L}_t^\star$. Viewing p_t as an element of the dual of $C_0^\infty(\mathbb{R}^d)$, (5) rewrites : for $f \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}), \qquad \int f(y) \frac{dp}{dt}(dy) = \int p_t(dy) \mathcal{L}_t f(y).$$

We have

$$\forall f \in C_0^\infty(\mathbb{R}^d), \forall t \le t' < T \quad < \frac{p_{t'} - p_t}{t' - t}, f > \overset{t' \to t}{\to} < p_t, \mathcal{L}_t f > = < \mathcal{L}_t^* p_t, f >,$$

where $< .,. >$ is the duality product.

For $z \in \mathbb{R}^d$, define the translation operator τ^z by $\tau_z f(x) = f(x+z)$. Then

$$
\begin{aligned}
& \int p_t(dx)\, \mathcal{L}_t f(x) \\
= & \int p_t(dx) \Big[b(t,x) \nabla f(x) + \frac{1}{2} \mathrm{tr}\left[\nabla^2 f(x)\, a(t,x) \right] \\
& \qquad + \int_{|z|>1} (\tau_z f(x) - f(x)) n(t,dz,x) \\
& \qquad + \int_{|z|\leq 1} (\tau_z f(x) - f(x) - z.\nabla f(x))\, n(t,dz,x) \Big] \\
= & \int \Big[- f(x) \frac{\partial}{\partial x}[b(t,x)p_t(dx)] + f(x) \frac{\partial^2}{\partial x^2}[\frac{a(t,x)}{2} p_t(dx)] \\
& \qquad + \int_{|z|>1} f(x)(\tau_{-z}(p_t(dx)n(t,dz,x)) - p_t(dx)n(t,dz,x)) \\
& \qquad + \int_{|z|\leq 1} f(x)(\tau_{-z}(p_t(dx)n(t,dz,x)) - p_t(dx)n(t,dz,x)) \\
& \qquad - z \frac{\partial}{\partial x}(p_t(dx)n(t,dz,x)) \Big],
\end{aligned}
$$

allowing to identify $\mathcal{L}^\star$. □

2.5 Martingale-preserving property

An important property of the construction of ξ in Theorem 2 is that it preserves the (local) martingale property: if ξ is a local martingale, so is X:

Proposition 2 (Martingale preserving property)**.**

1. *If ξ is a* local martingale *which satisfies the assumptions of Theorem 2, then its Markovian projection $(X_t)_{t\in[0,T]}$ is a local martingale on $(\Omega_0, \mathcal{B}_t, \mathbb{Q}_{\xi_0})$.*
2. *If furthermore*

$$\forall t \in [0,T] \quad \mathbb{E}^{\mathbb{P}}\left[\int \|y\|^2 \mu(dt\, dy) \right] < \infty,$$

then $(X_t)_{t\in[0,T]}$ is a square-integrable martingale.

Proof. 1) If ξ is a local martingale then the uniqueness of its semimartingale decomposition entails that

$$\beta_t + \int_{\|y\|\geq 1} y\, m(t,dy) = 0 \qquad dt \times \mathbb{P} - a.e.$$

hence $$\mathbb{Q}_{\xi_0}\left(\forall t \in [0,T], \quad \int_0^t ds \left[b(s,X_{s-}) + \int_{\|y\|\geq 1} y\, n(s,dy,X_{s-}) \right] = 0 \right) = 1.$$

The assumptions on m, δ then entail that X, as a sum of an Ito integral and a compensated Poisson integral, is a local martingale.
2) If $\mathbb{E}^{\mathbb{P}}\left[\int \|y\|^2 \mu(dt\,dy)\right] < \infty$ then

$$\mathbb{E}^{\mathbb{Q}_{\xi_0}}\left[\int \|y\|^2 n(t, dy, X_{t-})\right] < \infty,$$

and the compensated Poisson integral in X is a square-integrable martingale. □

3 Mimicking a semimartingale driven by a Poisson random measure

The representation (2) is not the most commonly used in applications, where a process is constructed as the solution to a stochastic differential equation driven by a Brownian motion and a Poisson random measure

$$\forall t \in [0, T] \qquad \zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int \psi_s(y)\, \tilde{N}(ds\, dy), \quad (21)$$

where $\xi_0 \in \mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, β and δ are non-anticipative càdlàg processes, N is a Poisson random measure on $[0, T] \times \mathbb{R}^d$ with intensity $\nu(dy)\, dt$ where

$$\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu(dy) < \infty, \qquad \tilde{N} = N - \nu(dy)dt, \quad (22)$$

and the random jump amplitude $\psi : [0, T] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, where $\mathcal{P}$ is the predictable σ-algebra on $[0, T] \times \Omega$. In this section, we shall assume that

$$\forall t \in [0, T], \quad \psi_t(\omega, 0) = 0 \quad \text{and} \quad \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \left(1 \wedge \|\psi_s(., y)\|^2\right) \nu(dy)\, ds\right] < \infty.$$

The difference between this representation and (2) is the presence of a random jump amplitude $\psi_t(\omega, .)$ in (21). The relation between these two representations for semimartingales has been discussed in great generality in [10, 17]. Here we give a less general result which suffices for our purpose. The following result expresses ζ in the form (2) suitable for applying Theorem 2.

Lemma 1 (Absorbing the jump amplitude in the compensator)**.**

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int \psi_s(z)\, \tilde{N}(ds\, dz)$$

can be also represented as

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int y\, \tilde{M}(ds\, dy), \quad (23)$$

where M is an integer-valued random measure on $[0,T] \times \mathbb{R}^d$ with compensator $\mu(\omega, dt, dy)$ given by

$$\forall A \in \mathcal{B}(\mathbb{R}^d - \{0\}), \qquad \mu(\omega, dt, A) = \nu(\psi_t^{-1}(\omega, A))\, dt,$$

where $\psi_t^{-1}(\omega, A) = \{z \in \mathbb{R}^d, \psi_t(\omega, z) \in A\}$ denotes the inverse image of A under the partial map ψ_t.

Proof. The result can be deduced from [10, Théorème 12] but we sketch here the proof for completeness. A Poisson random measure N on $[0,T] \times \mathbb{R}^d$ can be represented as a counting measure for some random sequence (T_n, U_n) with values in $[0,T] \times \mathbb{R}^d$

$$N = \sum_{n \geq 1} 1_{\{T_n, U_n\}}. \tag{24}$$

Let M be the integer-valued random measure defined by:

$$M = \sum_{n \geq 1} 1_{\{T_n, \psi_{T_n}(.,U_n)\}}. \tag{25}$$

μ, the *predictable* compensator of M is characterized by the following property [16, Thm 1.8.]: for any positive $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable map $\chi : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^+$ and any $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$,

$$\mathbb{E}\left[\int_0^t \int_A \chi(s,y)\, M(ds\, dy)\right] = \mathbb{E}\left[\int_0^t \int_A \chi(s,y)\, \mu(ds\, dy)\right]. \tag{26}$$

Similarly, for $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$

$$\mathbb{E}\left[\int_0^t \int_B \chi(s,y)\, N(ds\, dy)\right] = \mathbb{E}\left[\int_0^t \int_B \chi(s,y)\, \nu(dy)\, ds\right].$$

Using formulae (24) and (25):

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \int_A \chi(s,y)\, M(ds\, dy)\right] &= \mathbb{E}\left[\sum_{n \geq 1} \chi(T_n, \psi_{T_n}(., U_n))\right] \\
&= \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(.,A)} \chi(s, \psi_s(., z))\, N(ds\, dz)\right] \\
&= \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(.,A)} \chi(s, \psi_s(., z))\, \nu(dz)\, ds\right]
\end{aligned}$$

Formula (26) and the above equalities lead to:

$$\mathbb{E}\left[\int_0^t \int_A \chi(s,y)\, \mu(ds\, dy)\right] = \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(.,A)} \chi(s, \psi_s(., z))\, \nu(dz)\, ds\right].$$

Since ψ is a predictable random function, the uniqueness of the predictable compensator μ (take $\phi \equiv Id$ in [16, Thm 1.8.]) entails

$$\mu(\omega, dt, A) = \nu(\psi_t^{-1}(\omega, A))\, dt. \tag{27}$$

Formula (27) defines a random measure μ which is a Lévy kernel

$$\int_0^t \int \left(1 \wedge \|y\|^2\right)\, \mu(dy\, ds) = \int_0^t \int \left(1 \wedge \|\psi_s(., y)\|^2\right)\, \nu(dy)\, ds < \infty.$$

$\square$

In the case where $\psi_t(\omega, .) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is invertible and differentiable, we can characterize the density of the compensator μ as follows:

Lemma 2 (Differentiable case)**.** *If the Lévy measure $\nu(dz)$ has a density $\nu(z)$ and if $\psi_t(\omega, .) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$-diffeomorphism, then ζ, given in* (21)*, has the representation*

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int y\, \tilde{M}(ds\, dy),$$

where M is an integer-valued random measure with compensator

$$m(\omega; t, y)\, dt\, dy = 1_{\psi_t(\omega, \mathbb{R}^d)}(y)\, \left|\det \nabla_y \psi_t\right|^{-1} (\omega, \psi_t^{-1}(\omega, y))\, \nu(\psi_t^{-1}(\omega, y))\, dt\, dy,$$

where $\nabla_y \psi_t$ denotes the Jacobian matrix of $\psi_t(\omega, .)$.

Proof. We recall from the proof of Lemma 1:

$$\mathbb{E}\left[\int_0^t \int_A \chi(s, y)\, \mu(ds\, dy)\right] = \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(., A)} \chi(s, \psi_s(., z))\, \nu(z)\, ds\, dz\right].$$

Proceeding to the change of variable $\psi_s(., z) = y$:

$$\begin{aligned} & \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(., A)} \chi(s, \psi_s(., z))\, \nu(z)\, ds\, dz\right] \\ = \ & \mathbb{E}\left[\int_0^t \int_A 1_{\{\psi_s(\mathbb{R}^d)\}}(y)\, \chi(s, y)\ \left|\det \nabla \psi_s\right|^{-1} (., \psi_s^{-1}(., y))\, \nu(\psi_s^{-1}(., y)) ds\, dy\right]. \end{aligned}$$

The density appearing in the right hand side is predictable since ψ is a predictable random function. The uniqueness of the predictable compensator μ yields the result. $\square$

Let us combine Lemma 2 and Theorem 2. To proceed, we make a further assumption.

Assumption 7. *The Lévy measure ν admits a density $\nu(y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and:*

$$\forall t\in[0,T]\,\exists K_2>0 \quad \int_0^t \int_{\|y\|>1} \left(1\wedge\|\psi_s(.,y)\|^2\right)\,\nu(y)\,dy\,ds < K_2 \text{ a.s.}$$

and

$$\lim_{R\to\infty}\int_0^T \nu\left(\psi_t\left(\{\|y\|\geq R\}\right)\right)\,dt = 0 \quad \text{a.s.}$$

Theorem 4. *Let (ζ_t) be an Ito semimartingale defined on $[0,T]$ by the given the decomposition*

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\,ds + \int_0^t \delta_s\,dW_s + \int_0^t\int \psi_s(y)\,\tilde{N}(ds\,dy),$$

where $\psi_t(\omega,.) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is invertible and differentiable with inverse $\phi_t(\omega,.)$, β and δ satisfy Assumption 4 and ν Assumption 7. Define

$$m(t,y) = 1_{\{y\in\psi_t(\mathbb{R}^d)\}}\,|\det\nabla\psi_t|^{-1}\,(\psi_t^{-1}(y))\,\nu(\psi_t^{-1}(y)), \tag{28}$$

$$\begin{aligned} \text{and} \qquad b(t,z) &= \mathbb{E}\left[\beta_t\,|\zeta_{t^-}=z\right], \\ a(t,z) &= \mathbb{E}\left[{}^t\delta_t\delta_t\,|\zeta_{t^-}=z\right], \\ j(t,y,z) &= \mathbb{E}\left[m(t,y)\,|\zeta_{t^-}=z\right]. \end{aligned} \tag{29}$$

If δ, m satisfy Assumptions 5-6 and Assumption 2 holds for (b,a,j), then the stochastic differential equation

$$X_t = \zeta_0 + \int_0^t b(u,X_u)\,du + \int_0^t \Sigma(u,X_u)\,dB_u + \int_0^t\int y\,\tilde{J}(du\ dy), \tag{30}$$

where (B_t) is an n-dimensional Brownian motion, J is an integer valued random measure on $[0,T]\times\mathbb{R}^d$ with compensator $j(t,dy,X_{t-})\,dt$, $\tilde{J}=J-j$ and $\Sigma:[0,T]\times\mathbb{R}^d\mapsto M_{d\times n}(\mathbb{R})$ is a continuous function such that ${}^t\Sigma(t,z)\Sigma(t,z) = a(t,z)$, admits a unique weak solution $((X_t)_{t\in[0,T]},\mathbb{Q}_{\zeta_0})$ whose marginal distributions mimick those of ζ:

$$\forall t\in[0,T] \quad X_t \overset{\text{d}}{=} \zeta_t.$$

Under $\mathbb{Q}_{\zeta_0}$, X is a Markov process with infinitesimal generator $\mathcal{L}$ given by (3).

Proof. We first use Lemma 2 to obtain the representation (23) of ζ:

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\,ds + \int_0^t \delta_s\,dW_s + \int_0^t\int y\,\tilde{M}(ds\,dy)$$

Then, we observe that

$$\begin{aligned} \int_0^t\int y\,\tilde{M}(ds\,dy) &= \int_0^t\int_{\|y\|\leq 1} y\,\tilde{M}(ds\,dy) + \int_0^t\int_{\|y\|>1} y\,[M(ds\,dy)-\mu(ds\,dy)] \\ &= \int_0^t\int_{\|y\|\leq 1} y\,\tilde{M}(ds\,dy) + \int_0^t\int_{\|y\|>1} y\,M(ds\,dy) - \int_0^t\int_{\|y\|>1} y\,\mu(ds\,dy), \end{aligned}$$

where the terms above are well-defined thanks to Assumption 7. Lemma 2 leads to:

$$\int_0^t \int_{\|y\|>1} y\,\mu(ds\,dy) = \int_0^t \int_{\|\psi_s(y)\|>1} \|\psi_s(.,y)\|^2\,\nu(y)\,dy\,ds.$$

Hence:

$$\begin{aligned}\zeta_t = \zeta_0 + &\left[\int_0^t \beta_s\,ds - \int_0^t \int_{\|\psi_s(y)\|>1} \|\psi_s(.,y)\|^2\,\nu(y)\,dy\,ds\right] + \int_0^t \delta_s\,dW_s \\ &+ \int_0^t \int_{\|y\|\le 1} y\,\tilde{M}(ds\,dy) + \int_0^t \int_{\|y\|>1} y\,M(ds\,dy).\end{aligned}$$

This representation has the form (2) and Assumptions 4 and 7 guarantee that the local characteristics of ζ satisfy the assumptions of Theorem 2. Applying Theorem 2 yields the result. □

4 Examples

We now give some examples of stochastic models used in applications, where Markovian projections can be characterized in a more explicit manner than in the general results above. These examples also serve to illustrate that the continuity assumption (Assumption 2) on the projected coefficients (b, a, n) in (15) can be verified in many useful settings.

4.1 Semimartingales driven by a Markov process

In many examples in stochastic modeling, a quantity Z is expressed as a smooth function $f : \mathbb{R}^d \to \mathbb{R}$ of a d-dimensional Markov process Z: $\xi_t = f(Z_t)$. We will show that in this situation our assumptions will hold as soon as Z has an infinitesimal generator whose coefficients satisfy Assumptions 1, 2 and 3. Consider a time-dependent integro-differential operator $L = (L_t)_{t\in[0,T]}$ defined, for $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, by

$$\begin{aligned} L_t f(z) = b_Z(t,z).\nabla f(z) + \sum_{i,j=1}^d \frac{\Sigma_{ij}(t,x)}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ + \int_{\mathbb{R}^d} [f(z+\psi(t,z,y) - f(z) - \psi(t,y,z).\nabla f(z)]\nu(y)dy, \end{aligned} \tag{31}$$

where $\Sigma : [0,T]\times\mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$, $b_Z : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ and $\psi : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ are measurable functions and ν is a Lévy density.
We assume that

$$\begin{aligned} &\psi(.,.,0) = 0 \quad \psi(t,z,.)\,\text{is a}\,\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d) - \text{diffeomorphism} \\ &\exists K_2 > 0\,\forall t\in[0,T]\,\forall z\in\mathbb{R}^d \;\; \int_0^t \int_{\{\|y\|\ge 1\}} \left(1\wedge\|\psi(s,z,y)\|^2\right)\,\nu(y)\,dy\,ds < K_2. \end{aligned}$$

(28) can then be expressed as

$$m_Z(t,y,z) = 1_{\{y\in\psi_t(\mathbb{R}^d)\}} \left|\det\nabla\psi\right|^{-1}(t,z,\psi^{-1}(t,z,y))\,\nu(\psi^{-1}(t,z,y)).$$

Consider the stochastic differential equation

$$\begin{aligned}\forall t\in[0,T]\quad Z_t = Z_0 + \int_0^t b_Z(u,Z_{u-})\,du + \int_0^t \Sigma(u,Z_{u-})\,dW_u\\ + \int_0^t\int \psi(u,Z_{u-},y)\,\tilde{N}(du\ dy),\end{aligned} \tag{32}$$

where (W_t) is an n-dimensional Brownian motion, N is a Poisson random measure on $[0,T]\times\mathbb{R}^d$ with compensator $\nu(y)\,dy\,dt$, $\tilde{N}$ the associated compensated random measure. Throughout this section we shall assume that (b_Z,Σ,m_Z) satisfy Assumptions 1, 2 and 3. Proposition 1 then implies that for any $Z_0\in\mathbb{R}^d$, the above SDE admits a weak solution $((Z_t)_{t\in[0,T]},\mathbb{Q}_{Z_0})$, unique in law. Under $\mathbb{Q}_{Z_0}$, Z is a Markov process with infinitesimal generator L. Assume furthermore that Z_t has a density q_t with respect to the Lebesgue measure on $\mathbb{R}^d$.

Consider now the process

$$\forall t\in[0,T]\quad \xi_t = f(Z_t)\quad f:\mathbb{R}^d\to\mathbb{R} \tag{33}$$

The aim of this section is to build in an explicit manner the Markovian Projection of ξ_t for a sufficiently large class of functions f. Before stating the main result, we start with an useful Lemma, which will be of importance when one wants to characterize the yielding Markovian projection of ξ_t.

Lemma 3. *Let Z be a $\mathbb{R}^d$-valued random variable with density $q(z)$ and $f\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R})$ such that*

$$\forall z\in\mathbb{R}^d,\quad \frac{\partial f}{\partial z_d}(z)\neq 0. \tag{34}$$

Define the function $F:\mathbb{R}^d\to\mathbb{R}$ such that $f(z_1,\cdots,z_{d-1},F(z)) = z_d$. Then for any measurable function $g:\mathbb{R}^d\to\mathbb{R}$ such that $\mathbb{E}\left[|g(Z)|\right]<\infty$ and any $w\in f(\mathbb{R}^d)$,

$$\begin{aligned}&\mathbb{E}\left[g(Z)|f(Z)=w\right] \qquad = \\ &\int_{\mathbb{R}^{d-1}} dz_1...dz_{d-1}\quad g(z_1,\cdots,z_{d-1},w)\frac{q(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{\left|\frac{\partial f}{\partial z_d}(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))\right|}.\end{aligned}$$

Proof. Consider the d-dimensional random variable $\kappa(Z)$, where κ is defined by

$$\kappa(z) = \left(z_1,\cdots,z_{d-1},f(z)\right),$$

and let us compute the law of $\kappa(Z)$.

$$(\nabla_z\kappa) = \begin{pmatrix} 1 & 0 & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0\\ \frac{\partial f}{\partial z_1} & \cdots & \frac{\partial f}{\partial z_{d-1}} & \frac{\partial f}{\partial z_d}\end{pmatrix}.$$

One observes that $|\det(\nabla_z \kappa)|(z) = \left|\frac{\partial f}{\partial z_d}(z)\right| > 0$. Hence κ is a $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$-diffeomorphism with inverse κ^{-1}.

$$\kappa(\kappa^{-1}(z)) = (\kappa_1^{-1}(z), \cdots, \kappa_{d-1}^{-1}(z), f(\kappa_1^{-1}(z), \cdots, \kappa_d^{-1}(z)) = z.$$

For $1 \leq i \leq d-1$, $\kappa_i^{-1}(z) = z_i$ and $f(z_1, \cdots, z_{d-1}, \kappa_d^{-1}(z)) = z_d$ that is $\kappa_d^{-1}(z) = F(z)$. Hence

$$\kappa^{-1}(z_1, \cdots, z_d) = (z_1, \cdots, z_{d-1}, F(z)).$$

Define $q_\kappa(z)\, dz$ the inverse image of the measure $q(z)\, dz$ under the partial map κ by

$$\begin{aligned} q_\kappa(z) &= 1_{\{\kappa(\mathbb{R}^d)\}}(z)\, |\det(\nabla_z \kappa^{-1})|(z)\, q(\kappa^{-1}(z)) \\ &= 1_{\{\kappa(\mathbb{R}^d)\}}(z) \left|\frac{\partial f}{\partial z_d}\right|^{-1} (z_1, \cdots, z_{d-1}, F(z))\, q(z_1, \cdots, z_{d-1}, F(z)). \end{aligned}$$

$q_\kappa(z)$ is the density of $\kappa(Z)$. So, for any $w \in f(\mathbb{R}^d)$,

$$\begin{aligned} \mathbb{E}\left[g(Z)|f(Z) = w\right] &= \mathbb{E}\left[g(Z)|\kappa(Z) = (z_1, \cdots, z_{d-1}, w)\right] \\ &= \int_{\mathbb{R}^{d-1}} dz_1 ... dz_{d-1} g(z_1, \cdots, z_{d-1}, w) \frac{q(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}{\left|\frac{\partial f}{\partial z_d}\right|(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}., \end{aligned}$$

□

We can now formulate the main result of this section:

Theorem 5. *Let $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ with bounded derivatives such that*

$$\forall z \in \mathbb{R}^d \quad \frac{\partial f}{\partial z_d}(z) \neq 0. \tag{35}$$

Define, for $w \in f(\mathbb{R}^d), t \in [0, T]$,

$$\begin{aligned} b(t, w) = \int_{\mathbb{R}^{d-1}} &\Big[\nabla f(.).b_Z(t, .) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(.)^t \Sigma(t, .)\Sigma(t, .)\right] \\ &+ \int_{\mathbb{R}^d} \left(f(. + \psi(t, ., y)) - f(.) - \psi(t, ., y).\nabla f(.)\right)\, \nu(y)\, dy, \Big](z_1, \cdots, z_{d-1}, w) \\ &\times \frac{q_t(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}{\left|\frac{\partial f}{\partial z_d}\right|(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}., \\ \sigma(t, w) = \Big[\int_{\mathbb{R}^{d-1}} &\|\nabla f(.)\Sigma(t, .)\|^2 (z_1, \cdots, z_{d-1}, w) \\ &\times \frac{q_t(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}{\left|\frac{\partial f}{\partial z_d}\right|(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}.\Big]^{1/2}. \end{aligned} \tag{36}$$

and the measure $j(t,du,w)$ defined on $\mathbb{R}-\{0\}$ by

$$j(t,[u,\infty[,w) = \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}^d} 1_{\{f(.+\psi(t,.,y))-f(.)\geq u\}}(z_1,\cdots,z_{d-1},w)\,\nu(y)\,dy \right) \times \frac{q_t(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{\left|\frac{\partial f}{\partial z_d}(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))\right|}. \tag{37}$$

for $u>0$ and analogously for $u<0$. Then the stochastic differential equation

$$\begin{aligned} X_t = \xi_0 &+ \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dB_s \\ &+ \int_0^t \int_{\|y\|\leq 1} y\,\tilde{J}(ds\,dy) + \int_0^t \int_{\|y\|>1} y\,J(ds\,dy), \end{aligned} \tag{38}$$

where (B_t) is a Brownian motion, J is an integer-valued random measure on $[0,T]\times\mathbb{R}$ with compensator $j(t,du,X_{t-})\,dt$, $\tilde{J}=J-j$, admits a weak solution $((X_t)_{t\in[0,T]},\mathbb{Q}_{\xi_0})$, unique in law, whose marginal distributions mimick those of ξ:

$$\forall t\in[0,T] \quad X_t \overset{\mathrm{d}}{=} \xi_t.$$

Under $\mathbb{Q}_{\xi_0}$, X is a Markov process with infinitesimal generator $\mathcal{L}$ given by

$$\begin{aligned} \forall f\in\mathcal{C}_0^\infty(\mathbb{R}), \mathcal{L}_t f(w) = b(t,w)f'(w) &+ \frac{\sigma^2(t,w)}{2}\,f''(w) \\ &+ \int_{\mathbb{R}^d} [f(w+u)-f(w)-uf'(w)]j(t,du,w). \end{aligned}$$

Proof. Applying Itô's formula to $f(Z_t)$ yields

$$\begin{aligned} \xi_t &= \xi_0 + \int_0^t \nabla f(Z_{s^-}).b_Z(s,Z_{s-})\,ds + \int_0^t \nabla f(Z_{s^-}).\Sigma(s,Z_{s-})dW_s \\ &+ \frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(Z_{s-})^t\Sigma(s,Z_{s-})\Sigma(s,Z_{s-})\right]\,ds + \int_0^t \nabla f(Z_{s^-}).\psi(s,Z_{s-},y)\ \tilde{N}(ds\,dy) \\ &+ \int_0^t\int_{\mathbb{R}^d} \left(f(Z_{s^-}+\psi(s,Z_{s-},y)) - f(Z_{s^-}) - \psi(s,Z_{s-},y).\nabla f(Z_{s^-})\right)\,N(ds\,dy) \\ &= \xi_0 + \int_0^t \Big[\nabla f(Z_{s^-}).b_Z(s,Z_{s-}) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(Z_{s-})^t\Sigma(s,Z_{s-})\Sigma(s,Z_{s-})\right] \\ &\qquad + \int_{\mathbb{R}^d} \left(f(Z_{s^-}+\psi(s,Z_{s-},y)) - f(Z_{s^-}) - \psi(s,Z_{s-},y).\nabla f(Z_{s^-})\right)\,\nu(y)\,dy\Big]\,ds \\ &+ \int_0^t \nabla f(Z_{s^-}).\Sigma(s,Z_{s-})dW_s + \int_0^t\int_{\mathbb{R}^d} \left(f(Z_{s^-}+\psi(s,Z_{s-},y)) - f(Z_{s^-})\right)\,\tilde{N}(ds\,dy). \end{aligned}$$

If Σ satisfies assumption (i) then $(B_t)_{t\in[0,T]}$ defined by

$$dB_t = \frac{\nabla f(Z_{t^-}).\Sigma(t,Z_{t-})W_t}{\|\nabla f(Z_{t-})\Sigma(t,Z_{t-})\|},$$

is a continuous local martingale with $[B]_t = t$. Lévy's theorem implies that B is a Brownian motion.
If $\Sigma \equiv 0$ and (ii) holds, then ξ_t is a pure-jump semimartingale. Define $\mathcal{K}_t$

$$\mathcal{K}_t = \int_0^t \int \Psi(s, Z_{s-}, y)\, \tilde{N}(ds\, dy),$$

with $\Psi(t, z, y) = \psi(t, z, \kappa_z(y))$ where

$$\begin{array}{rcl} \kappa_z(y) : \mathbb{R}^d & \to & \mathbb{R}^d \\ y & \to & (y_1, \cdots, y_{d-1}, f(z+y) - f(z)). \end{array}$$

Since for any $z \in \mathbb{R}^d$, $|\det \nabla_y \kappa_z|(y) = \left|\frac{\partial}{\partial y_d} f(z+y)\right| > 0$, one can define

$$\kappa_z^{-1}(y) = (y_1, \cdots, y_{d-1}, F_z(y)) \quad F_z(y) : \mathbb{R}^d \to \mathbb{R} \quad f(z + (y_1, \cdots, y_{d-1}, F_z(y))) - f(z) = y_d.$$

Considering ϕ the inverse of ψ that is $\phi(t, \psi(t, z, y), z) = y$, define

$$\Phi(t, z, y) = \phi(t, z, \kappa_z^{-1}(y)).$$

Φ corresponds to the inverse of Ψ and Φ is differentiable on $\mathbb{R}^d$ with image $\mathbb{R}^d$.
Define the random measure $m(t, z, y)\, dt\, dy$

$$\begin{aligned} m(t, z, y) &= |\det \nabla_y \Phi(t, z, y)|\ \nu(\Phi(t, z, y)) \\ &= \left|\det \nabla_y \phi(t, z, \kappa_z^{-1}(y))\right| \left|\frac{\partial f}{\partial y_d}(z + \kappa_z^{-1}(y))\right|^{-1} \nu(\phi(t, z, \kappa_z^{-1}(y))). \end{aligned}$$

Using Assumption 7,

$$\begin{aligned} & \int_0^t \int_{\|y\|>1} \left(1 \wedge \|\Psi(s, z, y)\|^2\right) \nu(y)\, dy\, ds \\ = & \int_0^t \int_{\|y\|>1} \left(1 \wedge \left(\psi^1(s, z, y)^2 + \cdots + \psi^{d-1}(s, z, y)^2 + (f(z + \psi(s, z, y)) - f(z))^2\right)\right) \nu(y)\, dy\, ds \\ \leq & \int_0^t \int_{\|y\|>1} \left(1 \wedge \left(\psi^1(s, z, y)^2 + \cdots + \psi^{d-1}(s, z, y)^2 + \|\nabla f\|^2 \|\psi(s, z, y)\|^2\right)\right) \nu(y)\, dy\, ds \\ \leq & \int_0^t \int_{\|y\|>1} \left(1 \wedge (2 \vee \|\nabla f\|^2) \|\psi(s, z, y)\|^2\right) \nu(y)\, dy\, ds \end{aligned}$$

is bounded. One may apply Lemma 2 and express $\mathcal{K}_t$ as $\mathcal{K}_t = \int_0^t \int y \tilde{M}(ds\, dy)$ where $\tilde{M}$ is a compensated integer-valued random measure on $[0, T] \times \mathbb{R}^d$ with compensator $m(t, Z_{t-}, y)\, dy\, dt$.
Extracting the d-th component of $\mathcal{K}_t$, one obtains the semimartingale decomposition of ξ_t on $[0, T]$

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dB_s + \int_0^t \int u\, \tilde{K}(ds\, du),$$

where

$$\begin{cases} \beta_t &= \nabla f(Z_{t^-}).b_Z(t,Z_{t-}) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(Z_{t-})^t \Sigma(t,Z_{t-})\Sigma(t,Z_{t-})\right] \\ & \quad + \int_{\mathbb{R}^d} \left(f(Z_{t^-} + \psi(t,Z_{t-},y)) - f(Z_{t^-}) - \psi(t,Z_{t-},y).\nabla f(Z_{t^-}) \right) \nu(y)\, dy, \\ \delta_t &= \|\nabla f(Z_{t-})\Sigma(t,Z_{t-})\|, \end{cases}$$

and K is an integer-valued random measure on $[0,T]\times\mathbb{R}$ with compensator $k(t,Z_{t-},u)\,du\,dt$ defined for all $z\in\mathbb{R}^d$ via

$$\begin{aligned} k(t,z,u) &= \int_{\mathbb{R}^{d-1}} m(t,(y_1,\cdots,y_{d-1},u),z)\,dy_1\cdots dy_{d-1} \\ &= \int_{\mathbb{R}^{d-1}} |\mathrm{det}\nabla_y \Phi(t,z,(y_1,\cdots,y_{d-1},u))|\ \nu(\Phi(t,z,(y_1,\cdots,y_{d-1},u)))\,dy_1\cdots dy_{d-1}, \end{aligned} \tag{39}$$

and $\tilde{K}$ its compensated random measure. In particular for any $u>0$,

$$k(t,z,[u,\infty[) = \int_{\mathbb{R}^d} 1_{\{f(z+\psi(t,z,y))-f(z)\geq u\}}\,\nu(y)\,dy. \tag{40}$$

Let us show that if (b_Z,Σ,ν) satisfy Assumptions 1, 2 and 3 then the triplet $(\delta_t,\beta_t,k(t,Z_{t-},u))$ satisfies the assumptions of Theorem 2. First, note that β_t and δ_t satisfy Assumption 4 since $b_Z(t,z)$ and $\Sigma(t,z)$ satisfy Assumption 1 and ∇f and $\nabla^2 f$ are bounded.

$$\begin{aligned} \int_0^t \int \left(1\wedge\|u\|^2\right)\,k(s,Z_{s-},u)\,du\,ds &= \int_0^t \int \left(1\wedge |f(Z_{s^-}+\psi(s,Z_{s-},y)) - f(Z_{s^-})|^2\right)\,\nu(y)\,dy\,ds \\ &\leq \int_0^t \int \left(1\wedge \|\nabla f\|^2 \|\psi(s,Z_{s-},u)\|^2\right)\,\nu(y)\,dy\,ds, \end{aligned}$$

is bounded by Assumption 7. Hence k satisfies Assumption 5.
As argued before, one sees that if Σ is non-degenerate then δ_t is. In the case $\delta_t\equiv 0$, for $t\in[0,T[, R>0, z\in B(0,R)$ and $g\in\mathcal{C}_0^0(\mathbb{R})\geq 0$, consider C and

$K_T > 0$ chosen via Assumption 3

$$\begin{aligned}
& k(t,z,u) \\
= \quad & \int_{\mathbb{R}^{d-1}} |\det \nabla_y \Phi(t,z,(y_1,\cdots,y_{d-1},u))| \; \nu(\Phi(t,z,(y_1,\cdots,y_{d-1},u)))\, dy_1 \cdots dy_{d-1} \\
= \quad & \int_{\mathbb{R}^{d-1}} \left|\det \nabla_y \phi(t,z,\kappa_z^{-1}(y_1,\cdots,y_{d-1},u))\right| \; \left|\frac{\partial f}{\partial y_d}(z+\kappa_z^{-1}(y_1,\cdots,y_{d-1},u))\right|^{-1} \\
& \qquad\qquad\qquad\qquad \nu(\phi(t,z,\kappa_z^{-1}(y_1,\cdots,y_d,u)))\, dy_1 \cdots dy_{d-1} \\
\geq \quad & \int_{\mathbb{R}^{d-1}} \left|\frac{\partial f}{\partial y_d}(z+\kappa_z^{-1}(y_1,\cdots,y_{d-1},u))\right|^{-1} \frac{C}{\|\kappa_z^{-1}(y_1,\cdots,y_{d-1},u)\|^{d+\beta}}\, dy_1 \cdots dy_{d-1} \\
= \quad & \int_{\mathbb{R}^{d-1}} \frac{C}{\|(y_1,\cdots,y_{d-1},u)\|^{d+\beta}}\, dy_1 \cdots dy_{d-1} \\
= \quad & \frac{1}{|u|^{d+\beta}} \int_{\mathbb{R}^{d-1}} \frac{C}{\|(y_1/u,\cdots,y_{d-1}/u,1)\|^{d+\beta}}\, dy_1 \cdots dy_{d-1} \\
= \quad & C' \frac{1}{|u|^{1+\beta}},
\end{aligned}$$

with $C' = \int_{\mathbb{R}^{d-1}} C\, \|(w_1,\cdots,w_{d-1},1)\|^{-1}\, dw_1 \cdots dw_{d-1}$.
Similarly

$$\begin{aligned}
& \int \left(1 \wedge |u|^\beta\right) \left(k(t,z,u) - \frac{C'}{|u|^{1+\beta}}\right) du \\
= \quad & \int \left(1 \wedge |u|^\beta\right) \int_{\mathbb{R}^{d-1}} \left|\frac{\partial f}{\partial y_d}(z+\kappa_z^{-1}(y_1,\cdots,y_{d-1},u))\right|^{-1} \\
& \qquad \Big[\left|\det \nabla_y \phi(t,z,\kappa_z^{-1}(y_1,\cdots,y_{d-1},u))\right| \; \nu(\phi(t,z,\kappa_z^{-1}(y_1,\cdots,y_{d-1},u))) \\
& \qquad\qquad - \frac{C}{\|\kappa_z^{-1}(y_1,\cdots,y_{d-1},u)\|^{d+\beta}} \Big]\, dy_1 \cdots dy_{d-1}\, du \\
= \quad & \int_{\mathbb{R}^d} \left(1 \wedge |f(z+(y_1,\cdots,y_{d-1},u)) - f(z)|^\beta\right) \\
& \qquad \Big(|\det \nabla_y \phi(t,z,(y_1,\cdots,y_{d-1},u))| \; \nu(\phi(t,z,(y_1,\cdots,y_{d-1},u))) \\
& \qquad\qquad - \frac{C}{\|(y_1,\cdots,y_{d-1},u)\|^{d+\beta}} \Big)\, dy_1 \cdots dy_{d-1}\, du \\
\leq \quad & \int_{\mathbb{R}^d} \left(1 \wedge \|\nabla f\| \|(y_1,\cdots,y_{d-1},u)\|^\beta\right) \Big(|\det \nabla_y \phi(t,z,(y_1,\cdots,y_{d-1},u))| \; \nu(\phi(t,z,(y_1,\cdots,y_{d-1},u))) \\
& \qquad\qquad - \frac{C}{\|(y_1,\cdots,y_{d-1},u)\|^{d+\beta}} \Big)\, dy_1 \cdots dy_{d-1}\, du
\end{aligned}$$

is also bounded. Similar arguments would show that

$$\lim_{\epsilon\to 0}\int_{|u|\le\epsilon}|u|^{\beta}\left(k(t,Z_{t-},u)-\frac{C}{|u|^{1+\beta}}\right)du=0\text{ a.s.}$$

$$\text{and}\quad \lim_{R\to\infty}\int_0^T k\left(t,Z_{t-},\{|u|\ge R\}\right)dt=0\text{ a.s.},$$

since this essentially hinges on the fact that f has bounded derivatives. Define as in Theorem 2

$$\begin{aligned} b(t,w) &= \mathbb{E}\left[\beta_t|f(Z_{t-})=w\right], \\ \sigma(t,w) &= \mathbb{E}\left[\delta_t^2|f(Z_{t-})=w\right]^{1/2}, \\ j(t,u,w) &= \mathbb{E}\left[k(t,Z_{t-},u)|f(Z_{t-})=w\right]. \end{aligned}$$

Applying Lemma 3, one can compute explicitly the conditional expectations above. For example,

$$\begin{aligned} b(t,w) &= \int_{\mathbb{R}^{d-1}}\Big[\nabla f(.).b_Z(t,.)+\frac{1}{2}\mathrm{tr}\left[\nabla^2 f(.)^t\Sigma(t,.)\Sigma(t,.)\right] \\ &\quad + \int_{\mathbb{R}^d}\left(f(.+\psi(t,.,y))-f(.)-\psi(t,.,y).\nabla f(.)\right)\nu(y)\,dy,\Big](z_1,\cdots,z_{d-1},w) \\ &\quad \times\frac{q_t(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{|\frac{\partial f}{\partial z_d}(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))|}. \end{aligned}$$

with $F:\mathbb{R}^d\to\mathbb{R}$ defined by $f(z_1,\cdots,z_{d-1},F(z))=z_d$. Since f is C^2 with bounded derivatives, (b_Z,Σ,ν) satisfy Assumption 2 and $(t,z)\to q_t(z)$ is continuous in (t,z) on $[0,T]\times\mathbb{R}^d$ (see Theorem 1) then $b(t,.)$ is continuous on $\mathbb{R}$ uniformly in $t\in[0,T]$. Hence Assumption 2 holds for b. Proceeding similarly, one can show that that Assumption 2 holds for σ and j so Theorem 2 may be applied to yield the result. □

4.2 Time changed Lévy processes

Models based on time–changed Lévy processes have been the focus of much recent work especially in mathematical finance [6]. Let L_t be a Lévy process, (b,σ^2,ν) be its characteristic triplet, N its jump measure. Define

$$X_t=L_{\Theta_t}\qquad \Theta_t=\int_0^t\theta_s ds,$$

where (θ_t) is a locally bounded $\mathcal{F}_t$-adapted positive cadlag process, interpreted as the rate of time change.

Theorem 6 (Markovian projection of time-changed Lévy processes)**.** *Let L be a scalar Lévy process with triplet (b,σ^2,ν) and let $\xi_t=L_{\Theta_t}$ with $\Theta_t=\int_0^t\theta_s ds$*

where $\theta_t > 0$ is a positive semimartingale.
Define

$$\forall t \in [0,T] \quad \forall z \in \mathbb{R} \qquad \alpha(t,z) = E[\theta_t | \xi_{t-} = z],$$

and suppose that $\alpha(t,.)$ is continuous on $\mathbb{R}^d$, uniformly in $t \in [0,T]$. Assume that

$$\lim_{R\to\infty} \nu\left(\{\|y\| \geq R\}\right) = 0$$

and either (i) or (ii) holds for $(\sigma, \theta_t\, \nu)$, then

- *(ξ_t) has the same marginals as (X_t) on $[0,T]$, defined as the weak solution of*

$$\begin{aligned} X_t = \xi_0 &+ \int_0^t \sigma\sqrt{\alpha(s,X_{s-})}dB_s + \int_0^t b\alpha(s,X_{s-})ds \\ &+ \int_0^t \int_{|z|\leq 1} z\tilde{J}(ds\,dz) + \int_0^t \int_{|z|>1} zJ(ds\,dz), \end{aligned}$$

  where B_t is a real-valued brownian motion, J is an integer-valued random measure on $[0,T]\times\mathbb{R}$ with compensator $\alpha(t,X_{t-})\,\nu(dy)\,dt$.

- *The marginal distribution p_t of ξ_t is the unique solution of the forward equation:*

$$\frac{\partial p_t}{\partial t} = \mathcal{L}_t^\star.\, p_t,$$

  where, $\mathcal{L}_t^*$ is given by

$$\begin{aligned} \mathcal{L}_t^\star g(x) &= -b\frac{\partial}{\partial x}\left[\alpha(t,x)g(x)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}[\alpha^2(t,x)g(x)] \\ &+ \int_{\mathbb{R}^d} \nu(dz)\left[g(x-z)\alpha(t,x-z) - g(x)\alpha(t,x) - 1_{\|z\|\leq 1} z.\frac{\partial}{\partial x}[g(x)\alpha(t,x)]\right], \end{aligned}$$

  with the given initial condition $p_0(dy) = \mu_0(dy)$ where μ_0 denotes the law of ξ_0.

Proof. Consider the Lévy-Ito decomposition of L:

$$L_t = bt + \sigma W_t + \int_0^t \int_{|z|\leq 1} z\tilde{N}(dsdz) + \int_0^t \int_{|z|>1} zN(dsdz).$$

Then ξ rewrites

$$\begin{aligned} \xi_t &= \xi_0 + \sigma W(\Theta_t) + b\Theta_t \\ &+ \int_0^{\Theta_t} \int_{|z|\leq 1} z\tilde{N}(ds\,dz) + \int_0^{\theta_t} \int_{|z|>1} zN(ds\,dz). \end{aligned}$$

$W(\Theta_t)$ is a continuous martingale starting from 0, with quadratic variation $\Theta_t = \int_0^t \theta_s ds$. The Dubins-Schwarz theorem (see [22, Theorem 1.7]) implies that one can pick Z a Brownian motion independent of W, such that

$$W(\Theta_t) \stackrel{d}{=} \int_0^t \sqrt{\theta_s} dZ_s.$$

Hence X_t is the weak solution of :

$$\begin{aligned} X_t = X_0 &+ \int_0^t \sigma \sqrt{\theta_s}\, dZ_s + \int_0^t b\theta_s\, ds \\ &+ \int_0^t \int_{|z|\leq 1} z\theta_s\, \tilde{N}(ds\, dz) + \int_0^t \int_{|z|>1} z\theta_s\, N(ds\, dz). \end{aligned}$$

Using the notations of Theorem 2,

$$\beta_t = b\,\theta_t, \quad \delta_t = \sigma\,\sqrt{\theta_t}, \quad m(t, dy) = \theta_t\,\nu(dy).$$

Since Assumptions 4 and 5 are satisfied and :

$$\begin{aligned} b(t,.) &= \mathbb{E}\left[\beta_t | \xi_{t^-} = .\right] = b\,\alpha(t,.), \\ \sigma(t,.) &= \mathbb{E}\left[\delta_t^2 | \xi_{t^-} = .\right]^{1/2} = \sigma\,\sqrt{\alpha(t,.)}, \\ n(t,B,.) &= \mathbb{E}\left[m(t,B) | \xi_{t^-} = .\right] = \alpha(t,.)\nu(B), \end{aligned}$$

are all continuous on $\mathbb{R}$ uniformly in t on $[0,T]$. One may apply Theorem 2. □

The impact of the random time change on the marginals can be captured by making the characteristics state dependent

$$(\ b\alpha(t, X_{t-}), \sigma^2\alpha(t, X_{t-}), \alpha(t, X_{t-})\nu\ )$$

by introducing the *same* adjustment factor $\alpha(t, X_{t-})$ to the drift, diffusion coefficient and Lévy measure. In particular if $\alpha(t,x)$ is affine in x we get an affine process [8] where the affine dependence of the characteristics with respect to the state are restricted to be colinear, which is rather restrictive. This remark shows that time-changed Lévy processes, which in principle allow for a wide variety of choices for θ and L, may not be as flexible as apparently simpler affine models when it comes to reproducing marginal distributions.

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